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The Arithmetic Basis of Special Relativity

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Abstract

Under relatively general particle and rocket frame motions, it is shown that, for special relativity, the basic concepts can be formulated and the basic properties deduced using only arithmetic. Particular attention is directed toward velocity, acceleration, proper time, momentum, energy, and 4-vectors in both space-time and Minkowski space, and to relativistic generalizations of Newton's second law. The resulting mathematical simplification is not only completely compatible with modern computer technology, but it yields dynamical equations that can be solved directly by such computers. Particular applications of the numerical equations, which are either Lorentz invariant or are directly related to Lorentz-invariant formulas, are made to the study of a relativistic harmonic oscillator and to the motion of an electric particle in a magnetic field.

1. *Introduction*

The current availability of high-speed computer technology has motivated the study of compatible, arithmetic models (Cadzow, 1970; Greenspan, 1974; Mehta, 1967; Miller et al., 1972; Pasta and Ulam, 1959; LaBudde and Greenspan, 1974). Recently, for example (Greenspan, 1974; LaBudde and Greenspan, 1974), it has been shown that symmetry and all the conservation laws of Newtonian dynamics have an arithmetic basis. The aim of the present paper is to show that special relativity also has an arithmetic basis in that symmetry, conservation of linear momentum, conservation of energy, the Einstein rest energy equation, and the direct relationship between the displacement vector and the momentum-energy vector can all be deduced using only arithmetic formulas for basic physical quantities. Moreover, we will show how to apply the resulting Lorentz invariant numerical formulas to the solution of problems that are not solvable by classical mathematical methodology.

2. *Basics*

For simplicity only, let us consider two Euclidean coordinate systems XYZ and $X'Y'Z'$ which at some initial time coincide. Let the $X'Y'Z'$ system, called the rocket frame, be in constant uniform motion with respect to the XYZ frame, called the lab frame. Let this constant relative velocity be $\mathbf{u} = (u_1, u_2, u_3)$.

For $t_0 = 0$, let an observer in the lab frame make observations at the successive times t_k, $k = 0,1,2, \ldots$. Using an identical, synchronized clock, let an observer in the rocket frame make observations at the times t_k', $k = 0,1,2, \ldots$, where t_k' on the rocket clock corresponds to t_k on the lab clock.

If particle P is at (x_k, y_k, z_k) in the lab frame at time t_k, while it is at (x_k', y_k', z_k') in the rocket frame at time t_k', then we call x_k, y_k, z_k, t_k the space-time coordinates of event (x_k, y_k, z_k, t_k) in the lab frame, and, correspondingly, call x_k', y_k', z_k', t_k' the space-time coordinates of event (x_k', y_k', z_k', t_k') in the rocket frame. The space-time coordinates of the lab and the rocket frames are related by the Lorentz transformation, which is a linear algebraic relationship given as follows. Let

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) = \mathbf{u}/c \tag{2.1}$$

$$u^2 = u_1^{\,2} + u_2^{\,2} + u_3^{\,2} = c^2(\beta_1^{\,2} + \beta_2^{\,2} + \beta_3^{\,2}) = c^2\beta^2 \tag{2.2}$$

$$\gamma = (1 - \beta^2)^{-1/2} \tag{2.3}$$

where c is the speed of light. Let

$$\mathbf{r}_k = \begin{pmatrix} x_k \\ y_k \\ z_k \\ t_k \end{pmatrix}, \qquad \mathbf{r}_k' = \begin{pmatrix} x_k' \\ y_k' \\ z_k' \\ t_k' \end{pmatrix} \tag{2.4}$$

Then the Lorentz transformation $\mathscr{L} = (\mathscr{L}_{ij})$ is given (Arzelies, 1966, p. 74) by

$$\mathbf{r}_k' = \mathscr{L}\mathbf{r}_k \tag{2.5}$$

where

$$(\mathscr{L}_{ij}) = \begin{pmatrix} 1 + \beta_1^{\,2}\dfrac{\gamma^2}{\gamma+1} & \beta_1\beta_2\dfrac{\gamma^2}{\gamma+1} & \beta_1\beta_3\dfrac{\gamma^2}{\gamma+1} & -c\beta_1\gamma \\ \beta_1\beta_2\dfrac{\gamma^2}{\gamma+1} & 1 + \beta_2^{\,2}\dfrac{\gamma^2}{\gamma+1} & \beta_2\beta_3\dfrac{\gamma^2}{\gamma+1} & -c\beta_2\gamma \\ \beta_1\beta_3\dfrac{\gamma^2}{\gamma+1} & \beta_2\beta_3\dfrac{\gamma^2}{\gamma+1} & 1 + \beta_3\dfrac{\gamma^2}{\gamma+1} & -c\beta_3\gamma \\ -\dfrac{\beta_1}{c}\gamma & -\dfrac{\beta_2}{c}\gamma & -\dfrac{\beta_3}{c}\gamma & \gamma \end{pmatrix} \tag{2.6}$$

The transformation (2.6) is convenient from the physical point of view. From the geometric point of view, a more convenient form can be given as follows. Let new coordinates, called Minkowski coordinates (Arzelies, 1966, Chap. X) be defined by

$$\begin{aligned} x_{1,k} = x_k, x_{2,k} = y_k, x_{3,k} = z_k, x_{4,k} = ict_k \\ x'_{1,k} = x'_k, x'_{2,k} = y'_k, x'_{3,k} = z'_k, x'_{4,k} = ict'_k \end{aligned} \tag{2.7}$$

If

$$\mathbf{R}_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{pmatrix}, \qquad \mathbf{R}'_k = \begin{pmatrix} x'_{1,k} \\ x'_{2,k} \\ x'_{3,k} \\ x'_{4,k} \end{pmatrix} \tag{2.8}$$

then the Lorentz transformation $L = (L_{ij})$ is given (Arzelies, 1966, p. 74) by

$$\mathbf{R}'_k = L\mathbf{R}_k \tag{2.9}$$

where

$$(L_{ij}) = \begin{pmatrix} 1 + \beta_1^2 \dfrac{\gamma^2}{\gamma + 1} & \beta_1\beta_2 \dfrac{\gamma^2}{\gamma + 1} & \beta_1\beta_3 \dfrac{\gamma^2}{\gamma + 1} & i\beta_1\gamma \\ \beta_1\beta_2 \dfrac{\gamma^2}{\gamma + 1} & 1 + \beta_2^2 \dfrac{\gamma^2}{\gamma + 1} & \beta_2\beta_3 \dfrac{\gamma^2}{\gamma + 1} & i\beta_2\gamma \\ \beta_1\beta_3 \dfrac{\gamma^2}{\gamma + 1} & \beta_2\beta_3 \dfrac{\gamma^2}{\gamma + 1} & 1 + \beta_3^2 \dfrac{\gamma^2}{\gamma + 1} & i\beta_3\gamma \\ -i\beta_1\gamma & -i\beta_2\gamma & -i\beta_3\gamma & \gamma \end{pmatrix} \tag{2.10}$$

With regard to (2.10), note that

$$\sum_{j=1}^{4} L_{ij}L_{kj} = \delta_{i,k} \tag{2.11}$$

$$\sum_{j=1}^{4} L_{ji}L_{jk} = \delta_{i,k} \tag{2.12}$$

where $\delta_{i,k}$ is the Kronecker δ, and that

$$L^T L = LL^T = I \tag{2.13}$$

where L^T is the transpose of L and I is the identity.

Note also that the classical relativistic implications of the Lorentz transformation, like time dilation and Lorentz contraction, are, of course, valid.

3. *Velocity, Acceleration, and Proper Time*

Let the forward difference operator Δ at time t_k be defined as usual by

$$\Delta F(k) = F(k+1) - F(k)$$

Assume that particle P is in motion in the lab frame and at time t_k is at (x_k, y_k, z_k). Then P's velocity $\mathbf{v}_k$ and acceleration $\mathbf{a}_k$ at time t_k are defined by

$$\mathbf{v}_k = \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta x_k}{\Delta t_k} \\ \dfrac{\Delta y_k}{\Delta t_k} \\ \dfrac{\Delta z_k}{\Delta t_k} \end{pmatrix}, \qquad \mathbf{a}_k = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ a_{3,k} \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta v_{1,k}}{\Delta t_k} \\ \dfrac{\Delta v_{2,k}}{\Delta t_k} \\ \dfrac{\Delta v_{3,k}}{\Delta t_k} \end{pmatrix} \tag{3.1}$$

By the principle of relativity, P's velocity $\mathbf{v}_k'$ and acceleration $\mathbf{a}_k'$ in the rocket frame at time t_k' are defined by

$$\mathbf{v}_k' = \begin{pmatrix} v_{1,k}' \\ v_{2,k}' \\ v_{3,k}' \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta x_k'}{\Delta t_k'} \\ \dfrac{\Delta y_k'}{\Delta t_k'} \\ \dfrac{\Delta z_k'}{\Delta t_k'} \end{pmatrix}, \qquad \mathbf{a}_k' = \begin{pmatrix} a_{1,k}' \\ a_{2,k}' \\ a_{3,k}' \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta v_{1,k}'}{\Delta t_k'} \\ \dfrac{\Delta v_{2,k}'}{\Delta t_k'} \\ \dfrac{\Delta v_{3,k}'}{\Delta t_k'} \end{pmatrix} \tag{3.2}$$

The respective magnitudes v_k, v_k', a_k, a_k' of $\mathbf{v}_k, \mathbf{v}_k', \mathbf{a}_k, \mathbf{a}_k'$ are defined in the customary way by

$$v_k^2 = v_{1,k}^2 + v_{2,k}^2 + v_{3,k}^2, \qquad v_k'^2 = v_{1,k}'^2 + v_{2,k}'^2 + v_{3,k}'^2 \tag{3.3}$$

$$a_k^2 = a_{1,k}^2 + a_{2,k}^2 + a_{3,k}^2, \qquad a_k'^2 = a_{1,k}'^2 + a_{2,k}'^2 + a_{3,k}'^2 \tag{3.4}$$

The quantity τ_k, defined in the lab frame by

$$\tau_k = (c^2 t_k^2 - x_k^2 - y_k^2 - z_k^2)^{1/2} \tag{3.5}$$

is invariant under $\mathscr{L}$ since

$$(c^2 t_k'^2 - x_k'^2 - y_k'^2 - z_k'^2) = (c^2 t_k^2 - x_k^2 - y_k^2 - z_k^2)$$

When

$$c^2 t_k^2 - x_k^2 - y_k^2 - z_k^2 > 0 \tag{3.6}$$

τ_k is called the proper time of event (x_k, y_k, z_k, t_k), and, throughout, we assume that (3.6) is valid for all k. The quantity $\delta\tau_k$, defined by

$$\delta\tau_k = [c^2(\Delta t_k)^2 - (\Delta x_k)^2 - (\Delta y_k)^2 - (\Delta z_k)^2]^{1/2} \tag{3.7}$$

is, similarly, an invariant of $\mathscr{L}$ and is called the proper time between successive events (x_k, y_k, z_k, t_k) and $(x_{k+1}, y_{k+1}, z_{k+1}, t_{k+1})$. Throughout, we assume that, in (3.7),

$$c^2(\Delta t_k)^2 - (\Delta x_k)^2 - (\Delta y_k)^2 - (\Delta z_k)^2 > 0 \tag{3.8}$$

or, equivalently, that $v_k < c$, since (3.8) implies

$$c^2 - \left(\frac{\Delta x_k}{\Delta t_k}\right)^2 - \left(\frac{\Delta y_k}{\Delta t_k}\right)^2 - \left(\frac{\Delta z_k}{\Delta t_k}\right)^2 = c^2 - v_k^2 > 0$$

Note that $\delta\tau_k \neq \Delta\tau_k$ and $\delta\tau_k \neq d\tau_k$. For later convenience, observe also that

$$\delta\tau_k = \Delta t_k [c^2 - v_k^2]^{1/2} = \Delta t'_k [c^2 - v'^2_k]^{1/2} \tag{3.9}$$

Finally, note that

$$v'_{j,k} = \frac{\mathscr{L}_{j1}v_{1,k} + \mathscr{L}_{j2}v_{2,k} + \mathscr{L}_{j3}v_{3,k} + \mathscr{L}_{j4}}{\mathscr{L}_{41}v_{1,k} + \mathscr{L}_{42}v_{2,k} + \mathscr{L}_{43}v_{3,k} + \mathscr{L}_{44}}, \qquad j = 1,2,3 \tag{3.10}$$

from which it follows that $\mathbf{v}_k$ does not transform into $\mathbf{v}'_k$ the way $\mathbf{r}_k$ transforms into $\mathbf{r}'_k$. This is the basis of the usual statement that $\mathbf{v}_k$ is a vector in space, but not in space-time.

In Minkowski coordinates, (3.5) can be rewritten as

$$\tau_k = \{\sum_{i=1}^{4} [-(x_{i,k})^2]\}^{1/2} \tag{3.11}$$

while (3.7) becomes

$$\delta\tau_k = \{\sum_{i=1}^{4} [-(\Delta x_{i,k})^2]\}^{1/2} \tag{3.12}$$

and we define 4-velocities, or world velocities, and 4-accelerations, or world accelerations, in the following way: In Minkowski space, any quantity that has four components and is given in the lab frame by, say,

$$W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

and in the rocket frame by, say,

$$W' = \begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{pmatrix}$$

is called a 4-vector if

$$W' = LW \tag{3.13}$$

The prototype 4-vector in Minkowski space is, of course, $\mathbf{R}_k$, given by (2.8).

Now, suppose particle P is in motion and in the lab frame it is at (x_k, y_k, z_k) at time t_k while in the rocket frame it is at (x'_k, y'_k, z'_k) at the corresponding time t'_k. At time t_k in the lab frame, we define P's Minkowski 4-velocity $\mathbf{V}_k$ and Minkowski 4-acceleration $\mathbf{A}_k$ by

$$\mathbf{V}_k = \begin{pmatrix} V_{1,k} \\ V_{2,k} \\ V_{3,k} \\ V_{4,k} \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta x_{1,k}}{\delta\tau_k} \\ \dfrac{\Delta x_{2,k}}{\delta\tau_k} \\ \dfrac{\Delta x_{3,k}}{\delta\tau_k} \\ \dfrac{\Delta x_{4,k}}{\delta\tau_k} \end{pmatrix}, \quad \mathbf{A}_k = \begin{pmatrix} A_{1,k} \\ A_{2,k} \\ A_{3,k} \\ A_{4,k} \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta V_{1,k}}{\delta\tau_k} \\ \dfrac{\Delta V_{2,k}}{\delta\tau_k} \\ \dfrac{\Delta V_{3,k}}{\delta\tau_k} \\ \dfrac{\Delta V_{4,k}}{\delta\tau_k} \end{pmatrix} \tag{3.14}$$

By the principle of relativity, and recalling that $\delta\tau_k$ is invariant, $\mathbf{V}'_k$ and $\mathbf{A}'_k$ are defined by

$$\mathbf{V}'_k = \begin{pmatrix} V'_{1,k} \\ V'_{2,k} \\ V'_{3,k} \\ V'_{4,k} \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta x'_{1,k}}{\delta\tau_k} \\ \dfrac{\Delta x'_{2,k}}{\delta\tau_k} \\ \dfrac{\Delta x'_{3,k}}{\delta\tau_k} \\ \dfrac{\Delta x'_{4,k}}{\delta\tau_k} \end{pmatrix}, \quad \mathbf{A}'_k = \begin{pmatrix} A'_{1,k} \\ A'_{2,k} \\ A'_{3,k} \\ A'_{4,k} \end{pmatrix} = \begin{pmatrix} \dfrac{\Delta V'_{1,k}}{\delta\tau_k} \\ \dfrac{\Delta V'_{2,k}}{\delta\tau_k} \\ \dfrac{\Delta V'_{3,k}}{\delta\tau_k} \\ \dfrac{\Delta V'_{4,k}}{\delta\tau_k} \end{pmatrix} \tag{3.15}$$

Direct computation with (3.13) reveals easily that both $\mathbf{V}_k$ and $\mathbf{A}_k$ are 4-vectors. The relationship between components of $\mathbf{v}_k$ and the first three components of $\mathbf{V}_k$ can be established readily from (3.1), (3.9), and (3.14). Similar connections can be established between $\mathbf{a}_k$ and the first three components of $\mathbf{A}_k$.

The magnitude V_k of $\mathbf{V}_k$ is defined by

$$V_k^2 = \sum_{j=1}^{4} V_{j,k}^2 \tag{3.16}$$

An analogous definition holds also for $(V'_k)^2$. Note that (3.12) and (3.16) imply

$$V_k^2 = -1 \tag{3.17}$$

Thus, since (3.17) is valid for all k, the concept of 4-velocity, though geometrically convenient, is more restrictive physically than the three-dimensional velocity concept given by (3.1).

For completeness, let us show finally that

$$(V_k')^2 = (V_k)^2$$

the validity of which follows since

$$\begin{aligned}
\sum_{j=1}^{4} (V_{j,k}')^2 &= \sum_{j=1}^{4} (V_{j,k}')(V_{j,k}') \\
&= \sum_{j=1}^{4} \left(\sum_{m=1}^{4} L_{jm} V_{m,k} \right) \left(\sum_{n=1}^{4} L_{jn} V_{n,k} \right) \\
&= \sum_{m=1}^{4} \sum_{n=1}^{4} \delta_{mn} V_{m,n} V_{n,k} \\
&= \sum_{j=1}^{4} V_{j,k}^2
\end{aligned}$$

Note that 4-vectors with respect to space-time coordinates x_k, y_k, z_k, t_k can also be defined easily merely by replacing L with $\mathscr{L}$ in (3.13).

4. *Momentum and Energy*

We proceed now under the assumption that, without the presence of an external force, the interaction of two particles conserves linear momentum. To be precise, let particle P of mass m be in motion in the lab frame. At time t_k, the linear momentum $\mathbf{p}_k$ of P is defined by

$$\mathbf{p}_k = m\mathbf{v}_k \tag{4.1}$$

Similarly, in the rocket frame, let

$$\mathbf{p}_k' = m'\mathbf{v}_k' \tag{4.2}$$

The validity of momentum conservation follows (Taylor and Wheeler, 1966, pp. 101-110) if we require that in the lab frame

$$m = \frac{cm_0}{(c^2 - v_k^2)^{1/2}} \tag{4.3}$$

and, at the corresponding time in the rocket frame,

$$m' = \frac{cm_0}{(c^2 - v_k'^2)^{1/2}} \tag{4.4}$$

where m_0 is a constant called the rest mass of P.

We continue then by assuming the validity of (4.3) and (4.4).

The total energy E of particle P of mass m is defined by

$$E = mc^2 \tag{4.5}$$

Extensive experimental evidence now exists (Feyman, 1963, p. 15-11) to support the validity of (4.5), and the usual formula for rest energy $E_0 = m_0c^2$ follows readily.

To establish a relationship between momentum, energy, and rest mass, note that (4.1) and (4.3) imply

$$E_0^2 = p_k^2c^2 + m_0^2c^4$$

where p_k is the magnitude of $\mathbf{p}_k$.

5. *The Energy Momentum 4-Vector*

Thus far we have not placed particular emphasis on any special units of measurement. In this connection, we will now be relatively more specific in the following way. Let

$$E^* = E/c^2 \tag{5.1}$$

be a normalized energy in the sense that the units of E^* are units of mass. Then, from (4.5) and (5.1),

$$E^* = m \tag{5.2}$$

Our present purpose is to show that the quantity

$$\begin{pmatrix} mv_{1,k} \\ mv_{2,k} \\ mv_{3,k} \\ E^* \end{pmatrix}$$

is a 4-vector, called the energy momentum vector, with respect to $\mathscr{L}$. To do this observe that, with the help of (3.9), (4.3), and (4.4), one has

$$\mathscr{L}\begin{pmatrix} mv_{1,k} \\ mv_{2,k} \\ mv_{3,k} \\ E^* \end{pmatrix} = \mathscr{L}\begin{pmatrix} mv_{1,k} \\ mv_{2,k} \\ mv_{3,k} \\ m \end{pmatrix} = \mathscr{L}\begin{pmatrix} \dfrac{cm_0\Delta x_k}{(c^2 - v_k^2)^{1/2}\,\Delta t_k} \\ \dfrac{cm_0\Delta y_k}{(c^2 - v_k^2)^{1/2}\,\Delta t_k} \\ \dfrac{cm_0\Delta z_k}{(c^2 - v_k^2)^{1/2}\,\Delta t_k} \\ \dfrac{cm_0}{(c^2 - v_k^2)^{1/2}} \end{pmatrix} =$$

$$\mathscr{L}\begin{pmatrix} \dfrac{cm_0}{\delta\tau_k}\Delta x_k \\ \dfrac{cm_0}{\delta\tau_k}\Delta y_k \\ \dfrac{cm_0}{\delta\tau_k}\Delta z_k \\ \dfrac{cm_0}{\delta\tau_k}\Delta t_k \end{pmatrix} = \begin{pmatrix} \dfrac{cm_0}{\delta\tau_k}\Delta x_k' \\ \dfrac{cm_0}{\delta\tau_k}\Delta y_k' \\ \dfrac{cm_0}{\delta\tau_k}\Delta z_k' \\ \dfrac{cm_0}{\delta\tau_k}\Delta t_k' \end{pmatrix} = \begin{pmatrix} m'v'_{1,k} \\ m'v'_{2,k} \\ m'v'_{3,k} \\ m' \end{pmatrix}$$

and the assertion is proved.

6. *Dynamics*

Next, we examine possible relativistic extensions of Newton's second law and the invariance [called symmetry by some authors, as in Feyman (1963), and covariance by others, as in Schwartz (1968)] of such extensions under the Lorentz transformation.

It is an unfortunate mathematical consequence of continuous special relativistic theory (Bergmann, 1942, pp. 103–104) that the simple Einstein generalization

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{6.1}$$

does not, in general, transform under $\mathscr{L}$ into

$$\mathbf{F}' = \frac{d}{dt'}(m'\mathbf{v}') \tag{6.2}$$

although, interestingly enough, if both the particle and the rocket frames move in the same direction, then, indeed, does (6.1) transform into (6.2). To resolve this failure of the principle of relativity with respect to (6.1), two approaches have been followed. First (Arzelies, 1966, p. 268; Schwartz, 1968, p. 63), one can proceed under the approximating assumption that if a rocket frame were attached to P, so that it can have accelerated motion, and if at time t the velocity of P is $\mathbf{v}$, then one can treat the rocket frame at time t as being instantaneously in uniform relative motion with velocity $\mathbf{v}$ with respect to the lab frame. Indeed, such an assumption is tacitly made in the "clock paradox" (Arzelies, 1966, p. 63). Second (Bergmann, 1942, pp. 103–104; Muirhead, 1973, p. 86; Synge, 1965, pp. 165–167), one can formulate equations of motion directly in Minkowski space.

To develop the arithmetic analogs of the concepts and results described above, we will assume in Minkowski space the dynamical difference equation

$$\mathbf{F}_k = \alpha_k m_k \mathbf{A}_k - \frac{\Delta(\alpha_k m_k)}{\delta\tau_k}\left(\frac{\mathbf{V}_{k+1}+\mathbf{V}_k}{2}\right), \qquad \alpha_k m_k = m_0 \tag{6.3}$$

and in space-time a relative projection (Synge, 1965, p. 167) of the form

$$\mathbf{F}_k^p = c^2\,[m_k \mathbf{A}_k^p - (\Delta m_k/\delta\tau_k)\mathbf{V}_k^p] \tag{6.4}$$

where c^2 has replaced α_k in (6.3), where mass m at time t_k is denoted by m_k for computational convenience, and where the superscript p denotes the dropping of the fourth component of the given quantity. Note that (6.3) is analogous to the expanded form

$$\mathbf{F} = m\mathbf{a} + \mathbf{v}\frac{dm}{dt}$$

of (6.1) except for the sign between the terms. However, one can redefine $\mathbf{V}_k$ readily to yield agreement of signs also.

Let us show first that (6.4) is invariant under $\mathscr{L}$ provided that P and the rocket frame have velocities in the same direction. To do this, let us choose the lab frame and rocket frame coordinates so that motions are in the X direction only. Our problem then is to show that

$$F_{1,k} = c^2\,[m_k A_{1,k} - (\Delta m_k/\delta\tau_k)V_{1,k}] \tag{6.5}$$

and

$$F'_{1,k} = c^2\,[m'_k A'_{1,k} - (\Delta m'_k/\delta\tau_k)V'_{1,k}] \tag{6.6}$$

imply that

$$F_{1,k} = F'_{1,k}$$

From (6.5), then,

$$F_{1,k} = (c^2/\delta\tau_k)\,[m_k V_{1,k+1} - m_{k+1}V_{1,k}]$$

Now, under the present assumptions, (3.9) is valid, that is,

$$\delta\tau_k = \Delta t_k[c^2 - v_k^2]^{1/2}$$

so that

$$m_k = \frac{c m_0 \Delta t_k}{\delta\tau_k}$$

Thus,

$$F_{1,k} = \frac{c m_k}{m_0 \Delta t_k}(m_k V_{1,k+1} - m_{k+1}V_{1,k})$$

$$= \frac{c^2 m_k}{(\delta\tau_k/\Delta t_k)\cdot(\delta\tau_{k+1}/\Delta t_{k+1})}\left[\frac{(\Delta x_{k+1}/\delta\tau_{k+1})(\delta\tau_{k+1}/\Delta t_{k+1}) - (\Delta x_k/\delta\tau_k)(\delta\tau_k/\Delta t_k)}{\Delta t_k}\right]$$

so that

$$F_{1,k} = \frac{c^2 m_k}{[(c^2 - v_k^2)(c^2 - v_{k+1}^2)]^{1/2}} \frac{\Delta v_{1,k}}{\Delta t_k} \tag{6.7}$$

Hence,

$$\begin{aligned}
F_{1,k} &= \frac{c^2 m_k}{[(c^2 - v_k^2)(c^2 - v_{k+1}^2)]^{1/2}} \frac{\Delta v_{1,k}}{\Delta t_k} \\
&= \frac{c^3 m_0}{(c^2 - v_k^2)(c^2 - v_{k+1}^2)^{1/2}} \frac{\Delta v_{1,k}}{\Delta t_k} \\
&= \frac{c^3 m_0}{(c^2 - v_k'^2)(c^2 - v_{k+1}'^2)^{1/2}} \frac{\Delta v'_{1,k}}{\Delta t'_k} \\
&= \frac{c^2 m'_k}{[(c^2 - v_k'^2)(c^2 - v_{k+1}'^2)]^{1/2}} \frac{\Delta v'_{1,k}}{\Delta t'_k} \\
&= F'_{1,k}
\end{aligned}$$

and the invariance is established.

Note that as $\Delta t_k \to 0$, (6.7) reduces to the special form

$$F = \frac{c^2 m}{c^2 - v^2} \frac{dv}{dt}$$

of

$$F = \frac{d}{dt}(mv)$$

In Minkowski space there is a basic problem in the study of (6.3), which will be written now as

$$\mathbf{F}_k = m_0 \mathbf{A}_k - \frac{\Delta m_0}{\delta \tau_k} \left(\frac{\mathbf{V}_{k+1} + \mathbf{V}_k}{2} \right) \tag{6.8}$$

Indeed, equations (3.7) and (6.8) constitute nine equations for the eight quantities $x_{j,\,k+1}$, $V_{j,\,k+1}$, $j = 1,2,3,4$, a type of complexity which did not exist when considering (6.4) in Cartesian three-space (Synge, 1965, p. 166). In Minkowski space, then, one is forced to generate another unknown quantity, and the only candidate is the rest mass m_0. So, for the present, we must continue under the assumption that m_0 depends on time through $\mathbf{F}_k$. Under this assumption, by taking inner products of both sides of (6.8) with $(\mathbf{V}_{k+1} + \mathbf{V}_k)/2$ and by using (3.14) and (3.17), one finds

$$\mathbf{F}_k \cdot \left(\frac{\mathbf{V}_{k+1} + \mathbf{V}_k}{2} \right) = -\frac{\Delta m_0}{\delta \tau_k} \left(\frac{\mathbf{V}_{k+1} + \mathbf{V}_k}{2} \right) \cdot \left(\frac{\mathbf{V}_{k+1} + \mathbf{V}_k}{2} \right) \tag{6.9}$$

If one then chooses $\mathbf{F}_k$ in such a manner that

$$\mathbf{F}_k \cdot \left(\frac{\mathbf{V}_{k+1} + \mathbf{V}_k}{2} \right) = 0 \tag{6.10}$$

then, from (6.9), one can always choose $\Delta m_0 = 0$. Thus, restricting attention to forces which satisfy (6.10) yields

$$\mathbf{F}_k = m_0 \mathbf{A}_k \tag{6.11}$$

which is covariant under the Lorentz transformation and is completely analogous in structure to Newton's equation of motion.

Condition (6.10) restricts attention to forces that are orthogonal to the average velocity of a particle in motion. In the limit, it requires the force to be orthogonal to the particle's instantaneous velocity. This is, of course, the case in the most important application of special relativistic mechanics, that is, to the study of the motion of a charged particle in an electromagnetic field.

Note, also, with regard to (6.11), that for the special case $\mathbf{F}_k \equiv \mathbf{0}$, one has

$$m_0 \frac{\mathbf{V}_{k+1} - \mathbf{V}_k}{\delta \tau_k} \equiv \mathbf{0} \tag{6.12}$$

so that

$$\mathbf{V}_k \equiv \mathbf{V}_0, \; k = 0,1,2, \ldots \tag{6.13}$$

But, (6.13) implies

$$\frac{\mathbf{R}_{k+1} - \mathbf{R}_k}{\delta \tau_k} = \mathbf{V}_0, \; k = 0,1,2, \ldots$$

so that

$$\mathbf{R}_k = \mathbf{R}_0 + \mathbf{V}_0 \sum_{j=0}^{k-1} \delta \tau_j, \qquad k = 0,1,2, \ldots$$

which is lineal in Minkowski space.

7. *Computer Examples*

Consider first a particle P whose motion is one-dimensional, say, along an X axis, and is governed by the particular equation

$$\frac{d}{dt}(mv) = -x, \qquad t > 0 \tag{7.1}$$

In anology with the Newtonian case, where m is constant, P is called a relativistic harmonic oscillator. We propose to study the initial value problem defined by (7.1) and

$$x(0) = x_0 = 0, \qquad v(0) = v_0 \tag{7.2}$$

To do this, let us first rewrite (7.1) in the equivalent form

$$\frac{c^2 m}{c^2 - v^2}\frac{dv}{dt} = -x, \qquad t > 0 \tag{7.3}$$

Then, for $\Delta t > 0$ and $t_k = k\Delta t$, $k = 0,1,2,\ldots$, we approximate (7.3) by the difference equation

$$\frac{c^2 m_k}{[(c^2 - v_k^2)(c^2 - v_{k+1}^2)]^{1/2}} \frac{v_{k+1} - v_k}{t_{k+1} - t_k} = -x_k \tag{7.4}$$

which is Lorentz invariant. Lab and rocket calculations are then related by (2.5).

In order to proceed with the computer implementation, let us first simplify our formulas by adopting the absolute units $m_0 = c = 1$. Using (7.4), we may then write the position and velocity formulas in the equivalent forms

$$x_{k+1} = x_k + (\Delta t)v_k \tag{7.5}$$

$$v_{k+1} = \frac{v_k - (\Delta t)x_k(1 - v_k^2)^{3/2}[1 + x_k^2\Delta t^2(1 - v_k^2)]^{1/2}}{1 + x_k^2\Delta t^2(1 - v_k^2)^2} \tag{7.6}$$

and generate the motion recursively from initial conditions (7.2). This was done for 30,000 time steps with $\Delta t = 10^{-4}$ for each of the cases $v_0 = 0.001$, 0.01, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9. The FORTRAN program is given in Appendix 1 of Greenspan (1975) and the total running time on the UNIVAC 1110 was under 2 min. Figure 1 shows the amplitude and period of the first complete oscillation for the case $v_0 = 0.001$. For such a relatively low velocity, the oscillator should behave like a Newtonian oscillator, and, indeed, this is the case, with the amplitude being 0.001 and, to two decimal places, the period being 6.28($\sim 2\pi$). Subsequent motion of this oscillator continues to show almost no change in amplitude or period. At the other extreme, Figure 2 shows the motion for $v_0 = 0.9$, which is relatively close to the speed of light.

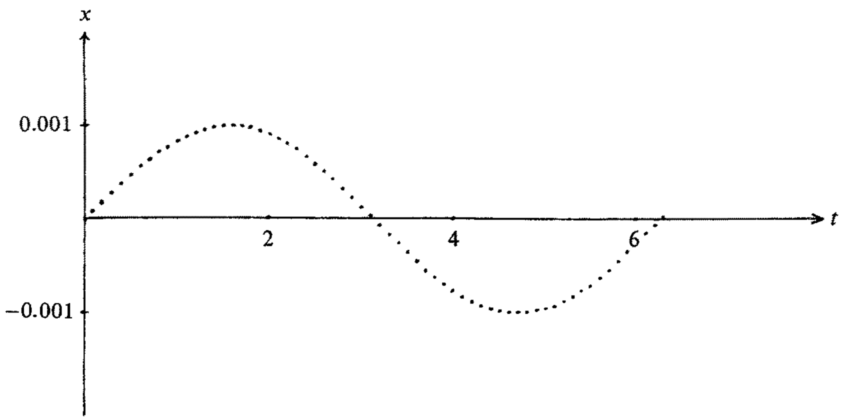

Figure 1

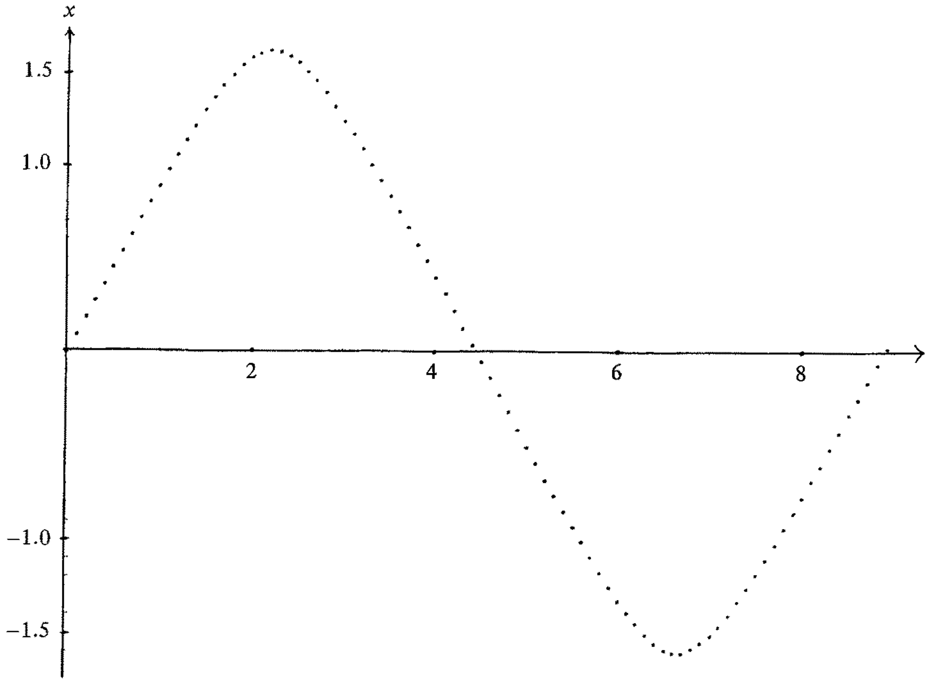

Figure 2

To two decimal places, the amplitude of the first oscillation is 1.61 while the period is 8.88. These results are distinctly non-Newtonian, and to 30,000 time steps, these results remain constant to two decimal places but do show small increments in the third decimal place. Finally, in Figure 3 is shown how the amplitude of a relativistic oscillator deviates from that of a Newtonian oscillator with increasing v_0.

We next turn to motion in more than one dimension. Consider, in particular, the motion of an electric charge e, moving in the X–Y plane under the influence of a magnetic field which acts in the direction of the Z axis. Assume that in the X–Y plane the force acting on the charge is

$$\mathbf{F} = (eHv_y, - eHv_x) \tag{7.7}$$

where v is the speed of the charge and H is the intensity of the field. The relativistic differential equations of motion are

$$\frac{d}{dt}(mv_x) = eHv_y \tag{7.8}$$

$$\frac{d}{dt}(mv_y) = -eHv_x \tag{7.9}$$

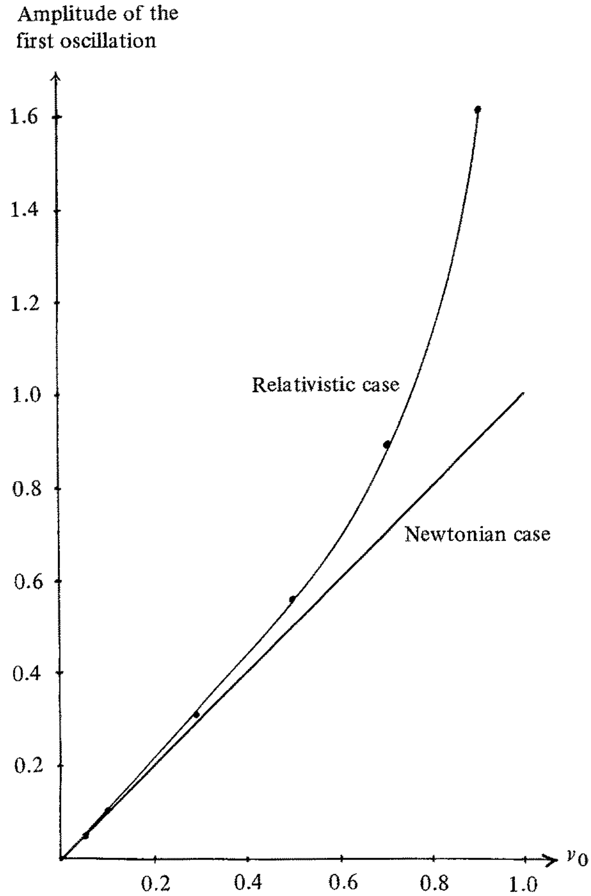

Figure 3

If H is uniform, then (7.8) and (7.9) can be solved analytically (Synge, 1965, p. 171) to yield circular motion. If H is not uniform, then, in general, (7.8) and (7.9) cannot be solved analytically.

Using absolute units $m_0 = c = e = 1$, let us begin with the general numerical approximations

$$F_{k,x} = \frac{m_k}{[(1 - v_k^2)(1 - v_{k+1}^2)]^{1/2}} \frac{v_{k+1,x} - v_{k,x}}{\Delta t_k},$$

$$F_{k,y} = \frac{m_k}{[(1 - v_k^2)(1 - v_{k+1}^2)]^{1/2}} \frac{v_{k+1,y} - v_{k,y}}{\Delta t_k}$$

or, equivalently,

$$F_{k,x} = \frac{v_{k+1,x} - v_{k,x}}{\Delta t_k(1 - v_k^2)(1 - v_{k+1}^2)^{1/2}}, \qquad F_{k,y} = \frac{v_{k+1,y} - v_{k,y}}{\Delta t_k(1 - v_k^2)(1 - v_{k+1}^2)^{1/2}} \tag{7.10}$$

where, of course,

$$x_{k+1} = x_k + v_{k,x}\Delta t_k, \qquad y_{k+1} = y_k + v_{k,y}\Delta t_k \tag{7.11}$$

Then (7.7) and (7.10) yield the following approximations of (7.8) and (7.9):

$$v_{k+1,x} - v_{k,x} - Hv_{k,y}(1 - v_{k,x}^2 - v_{k,y}^2)(1 - v_{k+1,x}^2 - v_{k+1,y}^2)^{1/2}\Delta t_k = 0 \tag{7.12}$$

$$v_{k+1,y} - v_{k,y} + Hv_{k,x}(1 - v_{k,x}^2 - v_{k,y}^2)(1 - v_{k+1,x}^2 - v_{k+1,y}^2)^{1/2}\Delta t_k = 0 \tag{7.13}$$

From (7.10), (7.12), and (7.13), one can construct readily, as described in Section 6, a related 4-force and a related set of Lorentz invariant dynamical difference equations in Minkowski 4-space. The numerical computations, however, are done more simply in Cartesian space using (7.11)-(7.13), so we continue to concentrate on these.

Let us consider the particular initial conditions

$$x_0 = y_0 = v_{0,x} = 0, \qquad v_{0,y} = 0.01 \tag{7.14}$$

For the parameter choices $\Delta t = 0.0001$ and $H = 100$, Figure 4 shows the resulting circular trajectory T_1 with center at $(0.0001, 0)$, radius $r = 0.0001$, and period $\tau = 0.063$, in complete agreement with the analytical solution (Synge, 1965, p. 171). Equations (7.12) and (7.13) are solved at each step by Newton's method with the velocity components at the previous time step being used to initiate the iteration. A comprehensive FORTRAN program for this example and for the one that follows is given in Appendix 2 of Greenspan (1975).

Consider next the initial value problem defined by (7.11)-(7.14), but in a nonuniform magnetic field with a "$1/r^2$" intensity given by

$$H = \frac{100}{1 + \alpha(x^2 + y^2)}, \qquad \alpha \geq 0 \tag{7.15}$$

Of course, for $\alpha = 0$, (7.15) reduces to the uniform case above, where $H = 100$. For the parameter choices $\Delta t = 0.0001$ and $\alpha = 10^7$, the resulting particle trajectory T_2 is shown also in Figure 4. The particle motion is initially similar to the circular motion of the first example, but as $(x^2 + y^2)$ increases and decreases, the varying effect of H results in the spiral type motion shown up to $t = 0.2$ in the figure.

Increasing the input parameter $v_{0,y}$ in both the above examples reveals quickly the price being paid for computational Lorentz invariance, for the

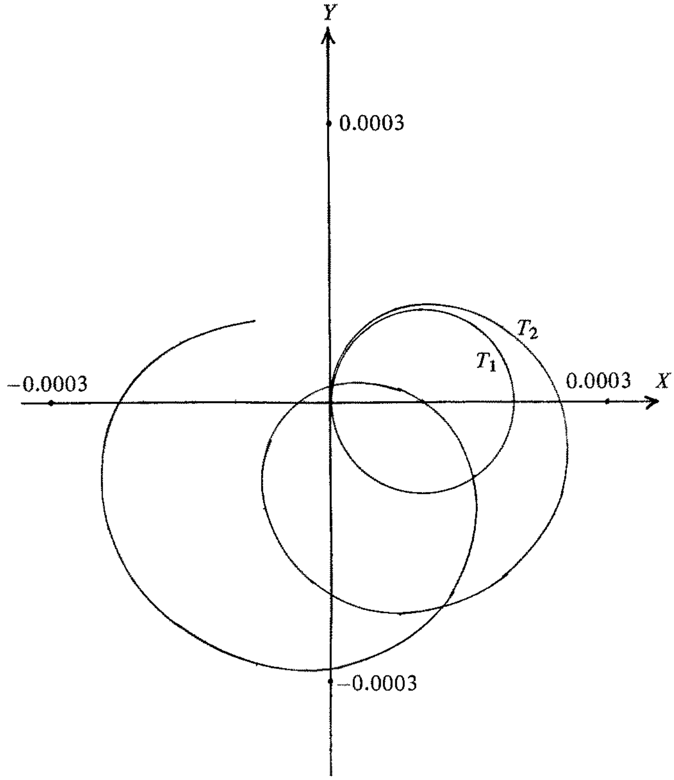

Figure 4

numerical formulas being used are of relatively low order and suffer from the usual shortcomings of such formulas. Thus, increasing $v_{0,y}$ to 0.1 in (7.14) results in having to reduce Δt to 10^{-7} to obtain reasonable accuracy on the UNIVAC 1110. Also, even for initial data (7.14), extended calculations with $\Delta t = 0.0001$ yield the inevitable, relatively large error accumulation associated with low-order methods. A most interesting and relevant question, then, which remains unanswered as yet, is whether or not there exist higher-order, Lorentz invariant numerical formulas.

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